ORIENTABLE 4-DIMENSIONAL POINCARÉ COMPLEXES HAVE REDUCIBLE SPIVAK FIBRATIONS

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ABSTRACT. We show that the Spivak normal fibration of an orientable 4-dimensional Poincaré complex has a vector bundle reduction.

1. Introduction

A Poincaré complex (*PD-complex*), as introduced by Wall [10, p. 214], is a (connected) finitely dominated CW complex X equipped with:

(i) a homomorphism $w\colon \pi_1(X) \to \{\pm 1\}$ defining a twisted $\Lambda := \mathbb{Z}\pi_1(X)$ module structure $\mathbb{Z}^t$ on $\mathbb{Z}$.
(ii) an integer n and a class $[X] \in H_n(X; \mathbb{Z}^t)$ such that
(iii) for all integers $r \geq 0$, cap product with $[X]$ induces an isomorphism

$$[X] \frown\colon H^r(X; \Lambda) \to H_{n-r}(X; \Lambda \otimes \mathbb{Z}^t) \ .$$

The integer $n = \dim X$ is called the *dimension* of X. It follows from the foundational results of Kirby and Siebenmann [5, Annex 3] that every closed topological n-manifold has the homotopy type of a Poincaré complex of dimension n (see the discussion in Wall [11, §17B]). In the manifold case, the homomorphism $w\colon \pi_1(X) \to \{\pm 1\}$ is given by the first Stiefel-Whitney class. Accordingly, a PD-complex X is called *orientable* if its homomorphism w is trivial.

Spivak [9] discovered that every simply-connected PD-complex X with $\dim X = n$ has an associated spherical fibration, denoted ν_X, which is unique up to stable fibre homotopy equivalence. It is constructed by embedding X in a high-dimensional Euclidean space $\mathbb{R}^{n+k}$ ($k \gg n$), and considering the fibration homotopic to the projection map $p\colon \partial N \to X$ from the boundary of a regular neighbourhood $N \subset \mathbb{R}^{n+k}$. The duality properties of X imply that the fibres of p are homotopy equivalent to S^{k-1}. The definition and the uniqueness statement were generalized by Wall [10, §3] to all PD-complexes, and ν_X is now called the *Spivak normal fibration* of X.

In the smooth manifold case, ν_X is the spherical fibration associated to the sphere bundle of the (stable) normal k-vector bundle of X. For topological manifolds, the corresponding notion is the (stable) normal $\mathbb{R}^k$-bundle ($k \gg n$), and its sub-bundle with fibres $\mathbb{R}^k - \{0\} \simeq S^{k-1}$.

After the further development of geometric surgery theory, due to Browder, Milnor, Novikov, Sullivan and Wall, the normal structures on PD-spaces and manifolds were re-expressed via classifying spaces (see [11, §10 and §17B], [5], [8], [6]). One outcome was

Date: August 15, 2018.
Research partially supported by NSERC Discovery Grant A4000.

the construction of a sequence of classifying spaces

$$BO \to BPL \to BTOP \to BG$$

relating smooth, PL, and topoogical bundles to spherical fibrations. In particular, the (stable) Spivak normal fibre space ν_X is classified by a map $\nu_X \colon X \to BG$.

Definition 1.1. We say that PD-complex X has a *reducible Spivak normal fibration* if the classifying map $\nu_X \colon X \to BG$ lifts to a map $\tilde{\nu}_X \colon X \to BTOP$.

Similarly, we say that the Spivak normal fibre space is reducible to a vector bundle if ν_X lifts to a map $\tilde{\nu}_X \colon X \to BO$. The lifting obstruction is given by the image of ν_X in $[X, B(G/TOP)]$ or $[X, B(G/O)]$, respectively. In dimensions ≥ 5, these are different problems, but if $\dim X \leq 4$ these two obstruction groups are the same because

$$[X, B(G/O)] = [X, B(G/PL)] = [X, B(G/O)] \cong H^3(X; \mathbb{Z}/2), \quad \text{if } \dim X \leq 4.$$

This is explained in Kirby-Taylor [6, §2]. In other words, the obstruction to reducibility for the Spivak normal fibration of a PD-complex X in dimensions ≤ 4 is a single characteristic class $k_3(X) \in H^3(X; \mathbb{Z}/2)$.

Theorem A. *Let X be an Poincaré complex. If $\dim X \leq 3$, or $\dim X = 4$ and X is orientable, then the Spivak normal fibration of X is reducible to a vector bundle.*

Remark 1.2. The dimension 4 case was known to the experts (see the statements in Spivak [9, p. 95] and Kirby-Taylor [6, p. 10]), but Land [7] pointed out the lack of a proof in the literature, and provided his own argument. For dimensions ≤ 2 the result is immediate, and the dimension 3 cases follow easily from the dimension 4 statement. In general, non-oriented PD-complexes in dimensions ≥ 4 do not have reducible Spivak normal fibrations (see Hambleton and Milgram [4] for explicit examples in every even dimension ≥ 4). The first non-reducible *orientable* example occurs in dimension 5 (see Gitler and Stasheff [3]).

Acknowledgement. I would like to thank Wolfgang Lück for asking me about this result at a conference in Regensburg (September, 2017). Andrew Ranicki and Larry Taylor later outlined alternate arguments, both different from the approach used by Markus Land, and different from the proof provided in this note.

2. The proof of Theorem A

Here is a short argument to show that an orientable 4-dimensional Poincaré complex has a reducible Spivak normal fibration. The proof is essentially contained in [4].

1. Suppose that X is an orientable 4-dimensional PD-complex such that ν_X is not reducible. Then by Poincaré duality there is a class $e \in H^1(X.\mathbb{Z}/2)$ such that

$$\langle k_3(X) \cup e, [X] \rangle \neq 0,$$

where $k_3(X)$ denotes the pullback to X of the first exotic characteristic class.

2. Let $f \colon X \to RP^\infty$ represent the cohomology class $e \in H^1(X; \mathbb{Z}/2)$. Then the element $0 \neq (X, f) \in \mathscr{N}_4^{PD}(RP^\infty)$ has Arf invariant $A(X, f) = 1$ (see [4], Corollary 4.2, Corollary 5.3, and Theorem 5.6).

3. By low-dimensional surgery, we may assume that $\pi_1(X) = \mathbb{Z}/2$ and that $f\colon X \to RP^\infty$ classifies its universal covering $\widetilde{X} \to X$ (see Wall [10, Corollary 2.3.2] to justify this much Poincaré surgery).

4. The form $B(a, b) = \langle a \cup T^*b, [X]\rangle$ is a symmetric unimodular bilinear form on $H^2(\widetilde{X}, \mathbb{Z})$, where T denotes the non-trivial covering involution. The form B is even (see Bredon [1, Chap VII, Theorem 7.4]).

5. The invariant $A(X, f)$ is the Arf invariant associated to the Browder-Livesay quadratic map q (see [2, §4], and [4, Theorem 1.4]), which refines the mod 2 reductions of B. By [2, Lemma 4.6], we have

$$q(a) \equiv \frac{B(a,a)}{2} \pmod 2$$

since $T\colon \widetilde{X} \to \widetilde{X}$ is orientation preserving. But B is an even unimodular symmetic bilinear form, so the Arf invariant obtained in this way is zero, and we have a contradiction. □

Remark 2.1. To obtain the reducibility results for 3-dimensional PD-complexes, one can make an appropriate circle bundle construction (which does not affect reducibility) resulting in orientable 4-dimensional PD-complexes, and then apply Theorem A.

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